Analytical results for random walk persistence

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Abstract

In this paper, we present the detailed calculation of the persistence exponent θ for a nearly-Markovian Gaussian process $X(t)$, a problem initially introduced in [*Phys. Rev. Lett.* **77**, 1420 (1996)], describing the probability that the walker never crosses the origin. Resummed perturbative and non-perturbative expressions for θ are derived, which suggest a connection with the result of the alternative independent interval approximation (IIA). The perturbation theory is extended to the calculation of θ for non-Gaussian processes, by making a strong connection between the problem of persistence and the calculation of the energy eigenfunctions of a quantum mechanical problem. Finally, we give perturbative and non-perturbative expressions for the persistence exponent $\theta(X_0)$, describing the probability that the process remains bigger than $X_0 \times \sqrt{\langle X^2(t)\rangle}$.

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I. INTRODUCTION

A natural quantity that characterizes a given stochastic process $X(t)$ is its persistence $P(t)$, *i.e.*, the probability that this signal has kept the same sign up to time t. For a large class of physical systems (to be defined more precisely below), persistence decays as a power-law in time, $P(t) \sim t^{-\theta}$ for large t, thus defining the persistence exponent θ.

This exponent has been studied in experimental systems (breath figures [1], a liquid crystal system mimicking the $2d$ Ising model [2], soap bubbles [3]...), and by theoretical means through the exact solution of models [4–7], numerical simulations [8,9], and general theoretical methods [10–15].

Most theoretical methods restrict themselves to the study of persistence of stochastic processes that are Gaussian. This is partly because Gaussian processes are abundant and simpler. Moreover, in many physical situations, the study of persistence of non-Gaussian signals can be effectively reduced to that of Gaussian signals [10,12]. Thus, given a Gaussian process $X(t)$ of zero mean, the basic question is: what is the probability that it remains, say, positive up to time t? This is a difficult problem that has been studied by mathematicians for a long time [16]. Recently, however, it has created much interest among physicists.

One of the general methods recently introduced to tackle this difficult problem, namely the independent interval approximation (IIA) [12], assumes that the interval lengths between zeros of the process $X(t)$ are statistically independent. This sole assumption permits the closure of a hierarchy of equations leading to an approximate expression of θ. This approximation gives very good results for smooth Gaussian processes (*i.e.* processes with a finite density of zero crossings). Unfortunately, it is not clear how this assumption can be justified, and whether a Gaussian process can be said *a priori* to be well described by this approximation.

An approximation for the distribution of the time-integrated "magnetization" [13,14], $M(t) = t^{-1} \int_0^t \mathrm{sign}(X(u))\, du$, also leads to good quantitative results for smooth processes, but suffers from the same conceptual problems as the IIA, and is not even guided by physical intuition. Still, the study of this quantity has lead to the introduction of a new quantity, the generalized persistence [13], that is the probability that $M(t)$ remains above a certain level M_0. This quantity which decays with a persistence exponent depending continuously on M_0 has been studied in the framework of spin systems and for random walkers [13].

Finally, a systematic ϵ-expansion, which is exact order by order, has been developed recently for smooth Gaussian processes [15].

However, all these approximate and exact techniques fail for processes that are singular, that is for which the density of zero crossings is infinite. These processes appear in many physical situations such as nearly Markovian random walkers [10] or interface growth [17].

In this paper, we come back to the first general method proposed, that is perturbation theory around a Gaussian and Markovian process [10]. After introducing the principle of this method (section **II** to **IV**), which shows a deep connection between the problems of persistence and the energy spectrum of a quantum mechanical problem, we present a symmetry argument for the exact form of θ which leads to more general results for the persistence exponent (section **V**). These results also reveal a connection between IIA and perturbative approaches. In section **VI**, we extend the perturbative approach to the case of non-Gaussian processes further reinforcing the link with standard quantum mechanics.

In section **VII**, we show that the various approaches introduced can be applied to the computation of the probability that the signal $X(t)/\sqrt{\langle X^2(t)\rangle}$ remains higher than any given non-zero-level X_0 (generalized persistence). Finally, in section **VIII**, we illustrate some of the results obtained in the preceding sections by means of numerical simulations.

II. IMPORTANCE OF GAUSSIAN STATIONARY PROCESSES

The most popular examples of persistent systems have been taken from the field of coarsening dynamics [18]. For instance, let us consider an Ising spin system after a quench at very low temperature from a high temperature disordered state. Domains of positive (essentially $+1$) and negative (essentially -1) magnetization grow with a time-dependent typical length scale $L(t) \sim t^{1/2}$. For this system, the spin persistence, that is, the probability that a spin has never changed sign, or has never been crossed by an interface, is known to decay as $t^{-\theta}$, with $\theta = 3/8$ in $d = 1$ [4], and $\theta \approx 0.22$ in $d = 2$ [8–10].

Due to dynamical scaling the two-time spin correlation function only depends on the dimensionless ratio of L at both considered times:

$$\langle S(t)S(t')\rangle = f(L(t)/L(t')). \tag{1}$$

This property will be characteristic of a coarsening system and only relies on the existence of a unique dynamical length scale and the dynamical scaling hypothesis [18]. Now, if $L(t)$ behaves as a power-law for large times, all two-point correlation functions are then functions of t/t'. By considering $\tau = \log(t)$, these correlation functions are then functions of $\exp|\tau - \tau'|$, or more simply $|\tau - \tau'|$, so that they become stationary in the fictitious time τ.

Moreover, in many physical systems [10,12,19], the question of computing the persistence for the original dynamical variable ($S(t)$ in the above example) can be reduced to the study of the persistence of a Gaussian variable $X(t)$. One possibility is, of course, that the physical variable is a Gaussian variable itself: this occurs in the study of the persistence of the diffusion equation [12], and for the total magnetization persistence of a spin system quenched at $T < T_c$ [20], or $T = T_c$ (in the latter case, the persistence exponent is a new critical exponent [19]). But in some other cases, including the Ising and more generally $O(N)$ spin systems, the original persistence can be shown to be very close to that of a true Gaussian process $X(t)$. For instance, a local spin in an Ising system behaves essentially as the sign of such a Gaussian process, $S(t) \approx \mathrm{sign}(X(t))$, an important result which was first used within the OJK theory [21,18], and later more precisely formalized by Mazenko and co-workers [22,18]. To summarize, we underline the special role played by Gaussian processes, and will thus restrict our study to this kind of process.

The next important remark is that if the persistence of the considered Gaussian process $X(t)$ decays as $t^{-\theta}$, the persistence in terms of the fictitious time τ (for which this process is stationary) is expected to decay exponentially as $\exp(-\theta\tau)$. Thus, in the following we restrict ourselves to the study of persistence for a stationary Gaussian process $X(\tau)$ [23–26]. Note that if $L(t)$ does not behave as a power-law of time, the persistence still decays as a power-law of $L(t)$ as soon as dynamical scaling is satisfied, and the proper fictitious time is simply $\tau = \log L(t)$, for which the process $X(\tau)$ is again stationary.

The most general equation of motion for a stationary Gaussian walker reads,

$$X'(\tau) = -\lambda X(\tau) + \int_{-\infty}^{\tau} J(\tau - \tau')\eta(\tau')d\tau', \tag{2}$$

where $\eta(\tau)$ is a Gaussian white noise satisfying $\langle \eta(\tau)\eta(\tau')\rangle = \delta(\tau-\tau')$. Indeed, this equation must be linear to preserve the Gaussian property, and the coefficient λ of $X(\tau)$ must be constant to preserve stationarity. The last term of Eq. (2) accounts for memory effects, involving a memory kernel J, and must take the form of a convolution product, again to preserve stationarity and the Gaussian property (linearity). Note that it is not necessary to involve higher derivatives of X in this equation of motion, as they can be accounted for by a proper choice of the kernel J (see Eq. (6) below).

The Markovian case is associated with $J(\tau) = \delta(\tau)$ (no memory effects), so that the equation of motion becomes,

$$X'(\tau) = -\lambda X(\tau) + \eta(\tau). \tag{3}$$

The velocity $X'(\tau)$ only involves the noise at the same time τ. For such a Langevin walker, the two-point correlation function is simply,

$$\langle X(\tau)X(\tau')\rangle = f(\tau - \tau'), \qquad f(\tau) = \frac{\exp -\lambda|\tau|}{2\lambda}. \tag{4}$$

For convenience, the correlator (and the variable X) has been normalized such that $f'(0^{\pm}) = \mp 1/2$, and, from now on, this will be assumed for all correlators. This will ensure that,

$$\omega^2 \hat{f}(\omega) \to 1, \quad \text{when} \quad \omega \to \pm\infty. \tag{5}$$

Also note that this Markovian correlator f has a cusp at the origin. We will define a *nearly* Markovian Gaussian process as one with a correlator which satisfy the above condition (5).

In general, the knowledge of the two-point correlation function $f(\tau)$ is equivalent to that of the equation of motion, as the Fourier transform of f satisfies,

$$\hat{f}(\omega) = \langle \hat{X}(\omega)\hat{X}(-\omega)\rangle = \frac{|\hat{J}(\omega)|^2}{\omega^2 + \lambda^2}. \tag{6}$$

This actually shows that any correlator $\hat{f}(\omega)$ can be reproduced by a proper (not unique) choice of the memory kernel $\hat{J}$.

In sections **III** and **IV**, we will give a more extensive account of the perturbative expansion for θ, in the case of a nearly Markovian Gaussian stationary process, a calculation which was first introduced in [10], and then reproduced in a real time formalism in [11]. This will be followed (section **IV**) by a resummation of this perturbation theory using a general symmetry argument, and the discovery of an intimate connection between the IIA and perturbative methods. A new non-perturbative expression for θ is also presented, which happens to reproduce quantitatively most numerical results (section **VIII**).

III. PERSISTENCE: THE MARKOVIAN CASE

Let us now move to the problem of persistence. The probability that a given walker remains on, say, the positive side of 0 at all times between 0 and β is,

$$P(\beta) = \frac{\int_{X>0} \mathcal{D}X(\tau) \exp[-\mathcal{S}]}{\int \mathcal{D}X(\tau) \exp[-\mathcal{S}]} = \frac{Z_1}{Z_0}, \tag{7}$$

where,

$$\mathcal{S}(\beta, \{X(\tau)\}) = \frac{1}{2} \int_0^\beta \int_0^\beta X(\tau_1) g(\tau_1 - \tau_2) X(\tau_2) \, d\tau_1 d\tau_2, \tag{8}$$

is the Gaussian weight associated with the trajectory $X(\tau)$, and $g(\tau_1 - \tau_2)$ is the inverse of the correlation matrix $f(\tau_1 - \tau_2)$. θ is then calculated from $P(\beta)$ by taking the limit,

$$\theta = - \lim_{\beta \to +\infty} \beta^{-1} \log P(\beta). \tag{9}$$

We can impose periodic boundary conditions for the walker trajectories, $X(0) = X(\beta)$, which should not affect the value of θ in the limit of large β. Indeed, in practice, the process will have a finite typical correlation time, equal to λ^{-1} in the example of the Markovian walker, so that this extra constraint cannot affect the large time persistence regime.

The path integrals of Eq. (7) strongly suggest the connection of this problem to Feynmann integrals in quantum mechanics or statistical field theory. Let us make this connection more precise. Because of the periodicity of the trajectories, the Gaussian weight in Eq. (7) can be also written,

$$\mathcal{S} = \frac{1}{2\beta} \sum_{n=0}^{+\infty} \hat{g}(\omega_n) |\hat{X}(\omega_n)|^2, \tag{10}$$

where $\hat{g}(\omega_n) = 1/\hat{f}(\omega_n)$ (the kernel in the expression of $\mathcal{S}$ is diagonal in Fourier space) and $\omega_n = 2\pi n/\beta$ are Matsubara frequencies. First consider a Markovian process for which $\hat{g}(\omega) = \omega^2 + \lambda^2$ (the Fourier transform of $f(\tau) = \exp(-\lambda|\tau|)/2\lambda$ is $[\omega^2 + \lambda^2]^{-1}$). $\mathcal{S}$ can be alternatively written as,

$$\mathcal{S} = \frac{1}{2} \int_0^\beta \left[\left(\frac{dX}{d\tau} \right)^2 + \lambda^2 X^2 \right] d\tau. \tag{11}$$

We recognize the action in imaginary time (β is then an inverse temperature) of an harmonic oscillator of frequency λ. The periodicity of the paths ensures that Z_0 $=\mathrm{Tr}\,[\exp -\beta H_0]$ is then the partition function of an harmonic oscillator, and Z_1 $=\mathrm{Tr}\,[\exp -\beta H_1]$, is the partition function of the same harmonic oscillator, but with an infinite wall at the origin (as the particle is constrained to remain on the positive axis). For large time, the persistence behaves as,

$$P(\beta) \sim \exp[-\beta(E_1 - E_0)], \tag{12}$$

where E_0 and E_1 are the ground state energies of these quantum systems. By direct identification, we thus find that,

$$\theta = E_1 - E_0. \tag{13}$$

Moreover, $E_0 = \lambda/2$, and it is easy to convince oneself that the ground state wavefunction of H_1, is the first excited state of H_0 restricted to the positive axis, so that $E_1 = 3\lambda/2$ (this argument is very general, and only relies on the $x \to -x$ symmetry of the potential). We finally find that $\theta = \lambda$ for a Markovian process. This is a well-known fact [23–26], that can be simply illustrated for the usual Langevin Markovian walker, for which the equation of motion reads (in actual time t), $\frac{dx}{dt} = \eta(t)$. For such a random walk, the persistence exponent is known to be 1/2 [23–26]. Let us reproduce this result within our approach. The two-point correlation function is easily computed: $\langle x(t)x(t')\rangle = \min(t,t')$, and the normalized variable $X(t) = x(t)/\sqrt{\langle x(t)^2\rangle}$, has a correlator, $\langle X(t)X(t')\rangle = (t'/t)^{1/2}$, for $t \geq t'$. This correlator is a function of the ratio of the two times, so that it is stationary after the change of variable $\tau = \log(t)$, becoming $\langle X(\tau)X(\tau')\rangle = \exp\left[-\frac{1}{2}|\tau-\tau'|\right]$. Applying the above calculation, we recover the result, $\theta = \lambda = 1/2$.

IV. PERTURBATION AROUND A GAUSSIAN MARKOVIAN PROCESS

Of course, this heavy machinery is not introduced to deal with the well understood Markovian case, but rather to be applied to the case of a nearly Markovian walker, for which no result exists. Thus, let us consider such a walker for which,

$$f(\tau) = \frac{1}{2\lambda}\left[\exp(-\lambda|\tau|) + \phi(\tau)\right], \tag{14}$$

where $\phi(\tau)$ is assumed to be a "small perturbation" to the Markovian correlator. In Fourier space this can be written, to first order in ϕ,

$$\hat{g}(\omega) = \hat{f}(\omega)^{-1} = \omega^2 + \lambda^2 - \hat{h}(\omega), \qquad \hat{h}(\omega) = \frac{(\omega^2+\lambda^2)^2}{2\lambda}\hat{\phi}(\omega). \tag{15}$$

In the general case, the denominator Z_0 of Eq. (7) can be exactly computed, as any unconstrained Gaussian integral, and is proportional to $\det^{1/2}[f(\tau_i - \tau_j)]$. After taking the proper limit, $E_0 = -\lim_{\beta\to+\infty}\beta^{-1}\log Z_0(\beta)$, one finds,

$$E_0 = -\frac{1}{2\pi}\int_0^{+\infty}\log\left(\omega^2\hat{f}(\omega)\right)\,d\omega. \tag{16}$$

Note that this integral converges thanks to the relation expressed in Eq. (5). To be consistent with the perturbative expansion for E_1 to come, we can write E_0 up to first order in ϕ,

$$E_0 = \frac{\lambda}{2} - \frac{1}{4\pi\lambda}\int_0^{+\infty}(\omega^2+\lambda^2)\hat{\phi}(\omega)\,d\omega + O(\phi^2), \tag{17}$$

the first term being the previously discussed Markovian result, that is the ground state energy of an harmonic oscillator of frequency λ. The computation of Z_1 (or E_1) is still a formidable task, as the domain of integration of the Gaussian integral only involves positive $X(\tau)$, for all τ. The natural impulse is to write $\mathcal{S} = \mathcal{S}_{osc.} + \delta\mathcal{S}$, where $\mathcal{S}_{osc.}$ is the harmonic oscillator action associated with a Markovian process (Eq. (11)), and,

$$
\begin{aligned}
\delta\mathcal{S} = & -\tfrac{1}{2}\int_0^\beta\int_0^\beta X(\tau_1)h(\tau_1-\tau_2)X(\tau_2)\,d\tau_1 d\tau_2, &(18)\\
= & -\tfrac{1}{2\beta}\sum_{n=0}^{+\infty}\hat{h}(\omega_n)|\hat{X}(\omega_n)|^2, &(19)
\end{aligned}
$$

where the Fourier transform of h is given in Eq. (15). $\delta\mathcal{S}$ is linear in ϕ and can be considered as a small perturbation to $\mathcal{S}_{osc.}$. We can now use the standard first order cumulant expansion of quantum mechanics (or statistical field theory), leading to,

$$
E_1 = \frac{3\lambda}{2} + \lim_{\beta\to+\infty}\langle\delta\mathcal{S}\rangle_{\text{wall}} + O(\phi^2), \tag{20}
$$

where the average is to be taken using the Boltzmann weight associated with the harmonic oscillator of frequency λ, with an infinite wall at the origin. Let us denote by $|\hat{l}\rangle$ the eigenstates of this quantum system (as opposed to $|l\rangle$, the eigenstates of the unconstrained oscillator), associated with the eigenenergies $\varepsilon_l = ((2l+1)+1/2)\lambda = (2l+3/2)\lambda$ ($l \geq 0$). One can then write,

$$
\langle\hat{0}|\hat{X}(-\omega_n)\hat{X}(\omega_n)|\hat{0}\rangle = \int_0^{+\infty} 2\cos\omega_n\tau \sum_{l=0}^{+\infty}|\langle\hat{0}|X|\hat{l}\rangle|^2 \mathrm{e}^{-(\varepsilon_l-\varepsilon_0)\tau}\,d\tau. \tag{21}
$$

$|\langle\hat{0}|X|\hat{l}\rangle|^2$ can be computed for the harmonic oscillator with a wall, using the fact that $\langle x|\hat{l}\rangle = \sqrt{2}\langle x|2l+1\rangle$, for $x \geq 0$, and exploiting standard properties of Hermite polynomials. The complete calculation is performed in appendix A and B. The final result reads,

$$
\langle\hat{0}|\hat{X}(-\omega_n)\hat{X}(\omega_n)|\hat{0}\rangle = \frac{8}{\lambda^2}\delta\left(\frac{\omega_n}{\lambda}\right) + \sum_{j=1}^{+\infty}\frac{4jc_j}{4j^2\lambda^2+\omega_n^2}, \tag{22}
$$

the Dirac peak coming from the $l = 0$ term. The coefficients c_j involved in this relation read,

$$
c_j = \frac{4}{\pi 2^{2j}(2j+1)!}\left(\frac{(2j)!}{j!(2j-1)}\right)^2. \tag{23}
$$

Finally, the sum over n in $\delta\mathcal{S}$ becomes an integral in the $\beta\to+\infty$ limit, leading to the final expression for $\theta = E_1 - E_0$:

$$
\theta = \lambda - \frac{1}{2\pi}\int_0^{+\infty}\hat{V}(\omega)\hat{\phi}(\omega)\,d\omega + O(\phi^2). \tag{24}
$$

The kernel $\hat{V}$ is defined by,

$$
\hat{V}(\omega) = \frac{(\omega^2+\lambda^2)^2}{2\lambda}\left[\frac{8}{\lambda}\delta(\omega) + \sum_{j=1}^{+\infty}\frac{4jc_j}{\omega^2+4j^2\lambda^2} - \frac{1}{\omega^2+\lambda^2}\right]. \tag{25}
$$

As noticed by Oerding *et al.*, this cumbersome expression in terms of Fourier transforms has a remarkably compact form when expressed in the inverse Fourier space. Indeed, the function between brackets is just the Fourier transform of,

$$
U(\tau) = \frac{1}{\lambda}\sum_{j=0}^{+\infty}c_j\exp(-2j\lambda|\tau|) - \frac{1}{2\lambda}\exp(-\lambda|\tau|), \tag{26}
$$

with $c_0 = \frac{4}{\pi}$, so that $\hat{V}$ is the Fourier transform of $\frac{1}{2\lambda}(-\partial_\tau^2 + \lambda^2)^2 U(\tau)$. This allows us to recast the preceding result into the form,

$$\theta = \lambda - \frac{1}{2\lambda} \int_0^{+\infty} \phi(\tau)(-\partial_\tau^2 + \lambda^2)^2 U(\tau)\, d\tau + O(\phi^2). \tag{27}$$

A simple manipulation on the c_j's (see appendix A) allows to resum exactly the series $(-\partial_\tau^2 + \lambda^2)^2 U(\tau)$, finally leading to,

$$\theta = \lambda \left[1 - \frac{2\lambda}{\pi} \int_0^{+\infty} \phi(\tau)[1 - \exp(-2\lambda\tau)]^{-3/2}\, d\tau\right] + O(\phi^2). \tag{28}$$

We can generalize this expression when the constraint on $X(\tau)$ is $X(\tau) \geq X_0$, instead of $X(\tau) \geq 0$ [13,27]. Indeed, for the Brownian walker ($f(\tau) = \exp(-|\tau|/2)$, such that $\langle X^2 \rangle = f(0) = 1$), it is known (see section **VII**) that θ satisfies $\mathcal{D}_{2\theta}(X_0) = 0$ [27], where $\mathcal{D}_{2\theta}$ is a parabolic cylinder function. We can expand this expression for small X_0, leading to $\theta_{\mathrm{Brownian}} = 1/2 + X_0/\sqrt{2\pi} + O(X_0)^2$. If we perturb around a general Markovian process ($f(\tau) = \frac{1}{2\lambda}\exp(-\lambda|\tau|)$), we then get another perturbative contribution for the exponent θ (valid in the limit of small X_0), which should be added to the result of Eq. (28):

$$\delta\theta(X_0) = 2\lambda \times \frac{X_0}{\sqrt{2\pi\langle X^2\rangle}} + O(X_0)^2 = \lambda^{3/2} \cdot \frac{2X_0}{\sqrt{\pi}} + O(X_0)^2. \tag{29}$$

V. RESUMMATION: A SYMMETRY ARGUMENT

A. Resummation in time

Consider a Gaussian process of correlator f and persistence exponent θ. Let us assume that we have been able to resum all terms of the perturbative expansion which contain only one time integral. Very generally, one can thus write,

$$\theta = \int_0^{+\infty} A(f(\tau)/f(0), \tau)\, d\tau. \tag{30}$$

The variable $f(\tau)/f(0)$ appears due to the fact that θ should not depend on the correlator normalization (here $f'(0^\pm) = \mp 1/2$, but $f(0) = 1$ was chosen in [11,12]).

If $f(\tau)$ is changed into $f(\alpha\tau)$, it is clear that the persistence exponent is simply changed into $\alpha\theta$. Using this remark, we get,

$$\alpha\theta = \int_0^{+\infty} A(f(\alpha\tau)/f(0), \tau)\, d\tau, \tag{31}$$

which shows after a simple change of variable that, for any process and any α, one must have,

$$\theta = \int_0^{+\infty} A(f(\tau)/f(0), \tau/\alpha)\, d\tau/\alpha^2. \tag{32}$$

This strongly suggests that θ can in fact be written as,

$$\theta = \int_0^{+\infty} B(f(\tau)/f(0)) \, \frac{d\tau}{\tau^2}. \tag{33}$$

Assuming now that $f(\tau)$ is close to a Markovian process with an associated small $\phi(\tau)$, one can develop Eq. (33) leading to,

$$\theta = \int_0^{+\infty} B(\exp(-\lambda\tau)) \, \frac{d\tau}{\tau^2} + \int_0^{+\infty} \phi(\tau) B'(\exp(-\lambda\tau)) \, \frac{d\tau}{\tau^2} + O(\phi^2). \tag{34}$$

In this perturbative limit, Eq. (34) should coincide with Eq. (28) leading to,

$$B'(\exp -X) = -\frac{2}{\pi} \frac{X^2}{(1 - \exp -2X)^{3/2}}, \tag{35}$$

or, after making the change of variable $u = \exp -X$,

$$B'(u) = -\frac{2}{\pi} \frac{\log(u)^2}{(1 - u^2)^{3/2}} \tag{36}$$

For the integral Eq. (33) to converge one should have $B(1) = 0$, finally leading to the final result of Eq. (33), with B given by:

$$B(u) = \frac{2}{\pi} \int_u^1 \frac{\log(v)^2}{(1 - v^2)^{3/2}} \, dv. \tag{37}$$

Note that this expression is not only defined for a nearly Markovian process, for which f has a cusp at $\tau = 0$, but actually converges for any process for which,

$$f(\tau)/f(0) - 1 \sim |\tau|^\mu, \quad \text{when} \quad \tau \to 0, \tag{38}$$

for any $\mu > 2/3$ (as $B(1 - \varepsilon) \sim \varepsilon^{3/2}$). Smooth processes (with a continuous velocity) are associated with $\mu = 2$, and the local density of a charge distribution evolving according to the simple diffusion (or heat) equation corresponds to this case [12]. As we have argued in detail, $\mu = 1$ corresponds to nearly Markovian processes. Finally, other values of $\mu < 2$ correspond to singular walkers for which the fractal density of the set of $X = 0$ crossing times is $1 - \mu/2$. Such processes have been encountered in the study of out of equilibrium atomic surfaces, for which $X(t)$ is the local height of the substrate [17].

A nice consistency check consists in showing that the first term in Eq. (34) is equal to λ, that is the Markovian value for θ. This is simply done by performing an integration by parts using the explicit expression of B, leading to,

$$\theta_{\text{Markov}} = \frac{2\lambda}{\pi} \int_0^{+\infty} \frac{(\lambda\tau)^2 \mathrm{e}^{-\lambda\tau}}{(1 - \mathrm{e}^{-2\lambda\tau})^{3/2}} \cdot \frac{d\tau}{\tau} = -\frac{2\lambda}{\pi} \int_0^1 \frac{\log u}{(1 - u^2)^{3/2}} \, du = \frac{2\lambda}{\pi} \int_0^1 \frac{du}{\sqrt{1 - u^2}} = \lambda, \tag{39}$$

the last integral being obtained through another integration by parts.

The argument presented above was motivated by the following important remark: for a given correlator f, the perturbation ϕ, or equivalently, the function $\exp(-\lambda|\tau|)/2\lambda$ around which the perturbation is started, are actually quite ill-defined. If we knew the complete

perturbation expansion, starting from any value of λ we should get the same result. Note that in standard field theory, one usually perturbs around a system which is solvable for a certain value (usually 0) of the coupling constant: there is a unique way of performing the perturbative expansion. Thus, it is natural to ask whether there is an optimal choice for the starting value of λ. A very natural choice is to take for λ the value which cancels the first order perturbative term. In other words, we take the "best" starting Markovian correlator such that the first order contribution vanishes. This gives another non-perturbative expression for θ (that we may call "variational" or self-consistent perturbative), which must satisfy,

$$\int_0^{+\infty} \frac{f(\tau)/f(0) - \exp(-\theta\tau)}{[1-\exp(-2\theta\tau)]^{3/2}}\, d\tau = 0. \tag{40}$$

This equation always has a solution, as the expression in Eq. (40) is clearly positive for $\theta \to +\infty$, and goes to $-\infty$, when $\theta \to 0$. The expression in Eq. (40) is defined for any $\mu > 1/2$, in fact a larger domain than the fully resummed formula of Eq. (37).

Note that we can write the resummed expression in a similar form,

$$\int_0^{+\infty} [B(f(\tau)/f(0)) - B(\exp(-\theta\tau)]\, d\tau = 0, \tag{41}$$

which after integration by parts, takes the form,

$$\int_0^{+\infty} \frac{K[f(\tau)/f(0), \exp(\theta\tau)]}{\left(1 - f^2(\tau)/f^2(0)\right)^{3/2}}\, d\tau = 0, \tag{42}$$

where the precise form of the known kernel K is of no real interest. This last remark allows us to make a link with the IIA result. Within this scheme, based on the approximation that the intervals between the zeros of the process are independent, it can be shown for smooth processes ($\mu = 2$) that θ must satisfy [12] (with the normalization $f(0) = 1$),

$$1 - \frac{\pi\theta}{2\sqrt{-f''(0)}} \cdot \left[1 + \frac{2\theta}{\pi}\int_0^{+\infty} \exp(\theta\tau)\sin^{-1}(f(\tau))\, d\tau\right] = 0. \tag{43}$$

If one integrates by parts this expression twice, it takes the following form:

$$\int_0^{+\infty} \exp(\theta\tau)\frac{\left[f''(1-f^2) + ff'^2\right](\tau)}{\left(1-f^2(\tau)\right)^{3/2}}\, d\tau = \sqrt{-2f''(0)}. \tag{44}$$

This expression now looks of the same type as the ones found within the perturbative approach. However, its domain of definition remains strictly $\mu = 2$.

Finally, let us mention that when the constraint on $X(\tau)$ is $X(\tau) \geq X_0$ [13,27] instead of $X(\tau) \geq 0$, the following perturbative correction should be added to the preceding expressions for θ (see Eq. (29)):

$$\delta\theta(X_0) = \theta(X_0 = 0) \cdot \frac{2X_0}{\sqrt{2\pi f(0)}} + O(X_0)^2. \tag{45}$$

B. Resummation in frequency space

The same argument as above can be applied to the expression of θ in frequency space. This time, this will allow us to resum all terms in the perturbation theory involving only one frequency integration. Again, we assume that θ can be written,

$$\theta = \int_0^{+\infty} C(\hat{f}(\omega), \omega)\, d\omega. \tag{46}$$

We still assume that f has a finite derivative in 0^+, keeping the normalization $2|f'(0^+)| = 1$.

If $f(\tau)$ is changed into $f(\alpha\tau)/\alpha$ (to preserve the normalization), $\hat{f}(\omega)$ is changed into $\hat{f}(\omega/\alpha)/\alpha^2$, and it is again clear that the persistence exponent is simply changed into $\alpha\theta$. Using this remark, we get:

$$\alpha\theta = \int_0^{+\infty} C\left(\alpha^{-2}\hat{f}(\omega/\alpha), \omega\right)\, d\omega, \tag{47}$$

which shows after a simple change of variable that, for any process and any α, one must have,

$$\theta = \int_0^{+\infty} C\left(\alpha^{-2}\hat{f}(\omega), \alpha\omega\right)\, d\omega. \tag{48}$$

This again strongly suggests that θ can in fact be written as,

$$\theta = \int_0^{+\infty} D(\omega^2\hat{f}(\omega))\, d\omega. \tag{49}$$

Note that this property is shared by the exact and general expression of E_0, given in Eq. (16).

Now, one can consider a nearly Markovian process, for which the correlator satisfies Eq. (15). One can develop Eq. (49) up to first order in $\hat{\phi}$ and identify the result to the perturbation result of Eq. (24-25). The calculation is elementary and leads to,

$$\theta = \frac{4}{\pi}\hat{f}^{-1/2}(0) + \frac{1}{2\pi}\int_0^{+\infty} \hat{W}(\omega^2\hat{f}(\omega))\, d\omega, \tag{50}$$

$$\hat{W}(x) = \sum_{n=1}^{+\infty} \frac{c_n}{n} \log\left(1 + 4n^2\left(x^{-1} - 1\right)\right) + \log(x). \tag{51}$$

The first term arises from the δ term in the kernel $\hat{V}$, and can also be written as $\frac{8}{\pi}\int_0^{+\infty} \delta\left(\omega|\hat{f}(\omega)|^{1/2}\right)\, d\omega$. Again, it is easy using relations given in appendix A to check that the Markovian value $\theta = \lambda$ is recovered for the Markovian correlator $\hat{f}(\omega) = (\omega^2 + \lambda^2)^{-1}$. Note finally that this procedure permits the recovery of the exact expression of E_0, which is produced by the last $\log(x)$ term in the kernel $\hat{W}$.

VI. PERTURBATION AROUND A NON-GAUSSIAN MARKOVIAN PROCESS

When writing $\mathcal{S} = \mathcal{S}_{osc.} + \delta\mathcal{S}$, we deliberately chose to perturb around a Gaussian Markovian walker, or around the quantum action of an harmonic oscillator in terms of the

pseudo-action $\mathcal{S}$. This is quite arbitrary, and in principle the action of any (preferably solvable) quantum system would have worked. The stochastic process associated to such an action (each trajectory $\{X(\tau)\}$ being weighted by $\exp -\mathcal{S}[\{X(\tau)\}]$) is Markovian but, in general, non-Gaussian, as the only Gaussian quantum action is that of an harmonic oscillator.

So in this section, we consider a stationary stochastic process $X(\tau)$ of any kind, associated with the weight or pseudo-action $\mathcal{S}$, and a quantum mechanical system for which the action is $\mathcal{S}_{\mathrm{Q}}$ ("Q" for "quantum"). This quantum system could be an harmonic oscillator, a particle in a square box, and more widely, any system preferably solvable for the actual perturbative calculation to be tractable.

Then, setting $\mathcal{S} = \mathcal{S}_{\mathrm{Q}} + \delta\mathcal{S}$, and reproducing exactly the calculations of the beginning of section **IV**, we end up with the following perturbative expression for θ:

$$\theta = E_1^{\mathrm{Q}} - E_0^{\mathrm{Q}} + \lim_{\beta\to+\infty} [\langle\delta\mathcal{S}\rangle_1 - \langle\delta\mathcal{S}\rangle_0] + O(\delta\mathcal{S}^2), \tag{52}$$

where E_0^{Q} (respectively E_1^{Q}) is the ground state energy of the unconstrained (respectively with an infinite wall at $X = 0$) quantum system. $\langle\quad\rangle_0$ and $\langle\quad\rangle_1$ denote quantum averages performed using the Hamiltonian of the quantum system, respectively without and with the infinite barrier at the origin.

We have already implicitly used Eq. (52) in section **IV**, where $\mathcal{S}_Q$ was chosen to be the Gaussian quantum action of an harmonic oscillator. Let us now illustrate Eq. (52) by taking a non-Gaussian system as the starting quantum system. The simplest possible example is that of particle in a box, with X restrained to the interval $[-b, b]$. We now use this simple non-Gaussian system to compute approximately the value of θ for a Gaussian process associated with the Gaussian weight $\mathcal{S}$ defined in Eq. (8).

Let us call $|l\rangle$ ($l \geq 0$), the eigenstates of a quantum particle in the box $[-b, b]$, associated with the eigenenergies $\varepsilon_l = \frac{1}{2}k_0^2(l+1)^2$, with $k_0 = \frac{\pi}{2b}$. The eigenstates of the constrained system (a particle in the box $[0, b]$) are the $|\hat{l}\rangle = \sqrt{2}|2l+1\rangle$ ($l \geq 0$). To evaluate $\langle\delta\mathcal{S}\rangle_1 - \langle\delta\mathcal{S}\rangle_0$, one essentially needs to compute $\langle 0|X(-\omega_n)X(\omega_n)|0\rangle$ and $\langle\hat{0}|X(-\omega_n)X(\omega_n)|\hat{0}\rangle$. This is a straightforward task using identities similar to Eq. (21), where the scalar products, $\langle 0|X|l\rangle$ and $\langle\hat{0}|X|\hat{l}\rangle$ are even easier to compute for a particle in a box (see appendix B). Introducing,

$$\hat{k}(\omega_n) = \langle 0|X(-\omega_n)X(\omega_n)|0\rangle = \frac{256}{\pi^2}\sum_{j=0}^{+\infty}\frac{a_j b_j}{k_0^4 b_j^2 + \omega_n^2}, \tag{53}$$

with,

$$a_j = \frac{2(j+1)^2}{(2j+1)^4(2j+3)^4}, \quad \text{and} \quad b_j = \frac{1}{2}(2j+1)(2j+3), \tag{54}$$

we get the following expressions for E_1 and E_0 ($\theta = E_1 - E_0$):

$$E_0 = \frac{k_0^2}{2} + \frac{1}{2\pi}\int_0^{+\infty}\left(\frac{\hat{k}(\omega)}{\hat{f}(\omega)} - 1\right)d\omega, \tag{55}$$

and,

$$E_1 = 2k_0^2 + \frac{\pi^2}{32}k_0^2\left(\frac{1}{k_0^4\hat{f}(0)} - \frac{12}{15-\pi^2}\right) \tag{56}$$

$$+\frac{1}{32\pi}\int_0^{+\infty}\left(\frac{1}{\hat{f}(\omega)} - \frac{1}{\hat{k}(\omega)}\right)\hat{k}(\omega/4)\,d\omega. \tag{57}$$

For a given correlator $\hat{f}(\omega)$, it is not clear what the "best" starting value for k_0 (or for the box size b) is. Let us propose two natural choices. We can take k_0 such that the first order perturbation vanishes, which leads to $\theta = 3k_0^2/2$. An alternative choice is to take k_0 such that E_1 is minimum, as it can be shown that $E_1(k_0)$ has always such a minimum for a finite k_0. In fact, the variational inequality $E_1 \le 2k_0^2 + \lim_{\beta\to+\infty}\langle\delta\mathcal{S}\rangle_1$ is exact for any k_0, which intuitively validates this choice if k_0.

VII. GENERALIZED PERSISTENCE

So far, we have essentially considered the probability that the signal $X(\tau)$ has never changed sign. In fact, it seems natural to study the more general probability that the signal has always remained above a certain level X_0. When $X_0 \neq 0$, this defines the X_0-level persistence. This generalized persistence has already been introduced for the simplest Markovian Gaussian walker [27], and spin systems [13]. Moreover, at least in the framework of Mazenko approximation [22], there is a connection between the persistence of the q-Potts model [5,6,9] (the probability that a given site always remains in a given phase) and the $X_0 = F(q)$-level persistence of a certain Gaussian variable [28].

Let us take the example of the Gaussian Markovian walker, associated, within our formalism, with the action of an harmonic oscillator. $E_1(X_0)$ is now the ground state energy of an harmonic oscillator with an infinite barrier at X_0, for which the eigenstates can be expressed in terms of a parabolic cylinder function [27] (generalization to a continuous index of Hermite polynomials). E_1 is then implicitly defined by imposing that the ground state eigenfunction has a unique node at $X = X_0$. If we come back to real time $t = \exp\tau$ (see section **III**), X_0-level persistence for the Langevin walker satisfying $\frac{dx}{dt} = \eta(t)$ is defined as the probability that $x(t)$ always remained greater than $X_0\sqrt{\langle x^2(t)\rangle} = X_0\sqrt{t}$. This decays as a power-law of time with exponent $\theta(X_0) = E_1(X_0) - E_0$.

If we were to compute the X_0-level persistence exponent $\theta(X_0)$ for a Gaussian process using the perturbation theory formalism, we would have to evaluate scalar products like $\langle\hat{0}|X|\hat{l}\rangle$, where $|\hat{l}\rangle$ are the eigenstates of the harmonic oscillator with an infinite barrier at $X = X_0$. Unfortunately, this seems to be an analytically untractable problem. However, using the formalism of the preceding section, we only have to evaluate brackets involving eigenstates of a particle in the box $[X_0, b]$, which are explicitly known. The calculation is straightforward (see appendix B) and leads to,

$$E_1(X_0) = \frac{2k_0^2}{(1-\eta)^2} + \frac{\pi^2}{32}k_0^2(1+\eta)^2\left(\frac{1}{k_0^4\hat{f}(0)} - \frac{12}{15-\pi^2}\right) \tag{58}$$

$$+\frac{k_0^2(1-\eta)^4}{32\pi}\int_0^{+\infty}\left(\frac{1}{\hat{f}(\omega)} - \frac{1}{\hat{k}(\omega)}\right)\hat{k}(\omega(1-\eta)^2/4)\,d\omega, \tag{59}$$

where $\eta = X_0/b$, and $\hat{k}$ has been defined in Eq. (53). As a check, we can see that for $\eta = 0$ (*i.e.* the wall is at the origin) we recover the result of Eq. (57), and for $\eta = -1$ (*i.e.* the wall is at $X_0 = -b$, which corresponds to no effective constraint), we recover the expression for E_0 of Eq. (55).

Again, for a given process X and a given level X_0, k_0 can be fixed by imposing that the first order perturbation term in θ vanishes, or by taking the value of k_0 for which $E_1(X_0, k_0)$ is minimum.

VIII. NUMERICAL SIMULATIONS

We now illustrate the various analytical results obtained in the preceding sections by means of numerical simulations.

A. Nearly Markovian processes

As already mentioned in the introduction, a local Ising spin evolving after a quench at $T = 0$, from the high temperature disordered phase, essentially behaves as the sign of a Gaussian variable. Mazenko approximation [22] then permits the calculation of the two-time correlator of this Gaussian process. It happens that, in one dimension, this approximation recovers the exact expression of $\langle S(t)S(t')\rangle$ [18], leading to the following form of the correlator f when expressed in the fictitious time $\tau = \log(t)$:

$$f(\tau) = \sqrt{\frac{2}{1 + \exp(\tau)}}. \qquad (60)$$

The exact value of θ in $d = 1$ is $\theta = 3/8 = 0.375$ [4]. The "variational" and resummed perturbative expression of Eq. (40) and Eq. (33-37) respectively lead to $\theta_{\text{var}} = .3595\ldots$ and $\theta_{\text{pert}} = .3677\ldots$. The process associated with the correlator given by Eq. (60) has been actually simulated using the Fourier space form of Eq. (2). We have obtained $\theta = 0.355 \pm 0.005$, in extremely good agreement with the theory. The small discrepancy with the exact result 0.375 for the Ising model is attributed to the fact that the actual process such that $S(t) = \text{sign}(X(t))$ is not strictly Gaussian. However it seems that this non-Gaussian effect is rather small.

We have also tested our theoretical expressions using a correlator introduced in [10]:

$$f(\tau) = \frac{2}{5}\exp(-\tau) + \frac{3}{5}\exp(-2\tau). \qquad (61)$$

We found $\theta_{\text{var}} = 1.4855\ldots$ and $\theta_{\text{pert}} = 1.4802\ldots$, again in good agreement with the numerical result $\theta = 1.481 \pm 0.005$.

B. Other singular processes

Interesting examples of singular correlators with $\mu < 2$ and $\mu \neq 1$ (see the definition of μ in section **V**) have been introduced in the framework of dynamical surfaces described by the following time-dependent equation [17]:

$$\frac{\partial h}{\partial t} = -(-\nabla^2)^{z/2} h + \eta, \tag{62}$$

where h is the local height of the fluctuating interface, and η is Gaussian white noise. The equation being linear, $h(x,t)$ is a Gaussian variable, for which we take the initial condition $h(x,0) = 0$. To define the first passage problems of interest, consider the quantity,

$$P(t_0, t) = \mathrm{Prob}[h(x,s) \neq h(x,t_0) \;\; \forall s : t_0 < s < t_0 + t], \tag{63}$$

and define θ_0 and θ_s as,

$$p_0(t) \equiv P(0,t) \sim t^{-\theta_0}, \quad t \to +\infty, \tag{64}$$

$$p_s(t) \equiv \lim_{t_0 \to +\infty} P(t_0, t) \sim t^{-\theta_S}, \quad t \to +\infty. \tag{65}$$

p_0 measures the first passage exponent of the growing interface, whereas p_s contains the relevant information, when the interface has entered the steady state ($t_0 \to +\infty$).

The correlators associated with these two persistence problems are respectively (when expressed as functions of the fictitious time τ) [17]:

$$f_0(T) = \cosh(\tau/2)^\mu - |\sinh(\tau/2)|^\mu \tag{66}$$

$$f_s(T) = \cosh(\mu\tau/2) - \tfrac{1}{2}|2\sinh(\tau/2)|^\mu, \tag{67}$$

and both satisfy $1 - f_{0,s}(\tau) \sim \tau^\mu$, for small τ, with $\mu = 1 - d/z$ ($\mu = 1 - (d+2)/z$ for a volume conserving noise). We now simply treat μ as a free parameter. Using a connection to the fractional Brownian walker, it has been conjectured that $\theta_s = 1 - \mu/2$ [17], which has been confirmed by numerical simulations.

Let us take two typical values for μ. For $\mu = 3/2$, we find $\theta_{0,\mathrm{var}} = 0.2088\ldots$ and $\theta_{0,\mathrm{pert}} = 0.2146\ldots$, which compare well to the numerical value $\theta_0 = 0.201 \pm 0.005$. For the case of the steady interface, the conjectured persistence exponent is $\theta_s = 1/4$, in good agreement with the simulations ($\theta_s = 0.247 \pm 0.005$). Variational and perturbative methods are reasonably accurate, giving $\theta_{\mathrm{s,var}} = 0.2583\ldots$ and $\theta_{\mathrm{s,pert}} = 0.2644\ldots$. Note that the first order perturbation expression of Eq. (28) reproduces exactly the conjectured value for θ_s.

We have also tested the case $\mu = 3/4$, which is getting dangerously close to the limit of the validity domain of our variational ($\mu_{\mathrm{var}} = 1/2$) and perturbative ($\mu_{\mathrm{pert}} = 2/3$) expressions. The numerical value of θ_0 is $\theta_0 = 0.85 \pm 0.01$, for which the variational approach gives $\theta_{0,\mathrm{var}} = 0.8852\ldots$. Not surprizingly, the resummed perturbation leads to a bad result ($\theta_{0,\mathrm{pert}} \approx 1.1$). The conjectured value for θ_s is $\theta_s = 0.625$, while the simulation of the process leads to $\theta_s = 0.625 \pm 0.005$, and that of the associated discrete solid-on-solid model leads to $\theta_s = 0.635 \pm 0.005$ (see [17] for details). We find a qualitatively correct value of $\theta_{\mathrm{s,var}} = 0.6662\ldots$, but the resummed perturbation fails again ($\theta_{\mathrm{s,pert}} \approx 0.84$).

C. Smooth processes

For singular processes ($\mu < 2$), it was not possible to compare our variational and perturbative results to the IIA expressions of Eq. (43-44), which are only defined for smooth processes.

One of the most spectacular example of smooth Gaussian processes has been given in [12]: consider an initially random spatial distribution of charges of zero average, $\rho(\mathbf{x}, t = 0)$. It then evolves according to the simple diffusion equation,

$$\frac{\partial \rho}{\partial t}(\mathbf{x}, t) = \nabla^2 \rho(\mathbf{x}, t). \tag{68}$$

The persistence is defined as the probability that the local charge at a given $\mathbf{x}$ never changes sign. It decays as a power-law, defining θ_d, the dimension-dependent persistence exponent.

The IIA [12] (as well as the specific method of [14]) is in amazing agreement with numerical simulations. For instance, in $d = 1$, $\theta_{\rm IIA} = 0.1203\ldots$, to be compared to the numerical value $\theta = 0.1207 \pm 0.0005$. The agreement seems to be of the same order in any dimension. Smooth processes are in principle beyond the range of application of perturbative methods. Still, the variational approach remains qualitatively correct, leading to $\theta_{\rm var} = 0.1428\ldots$ for the one-dimensional diffusion equation, whereas the resummed perturbation theory is again quite bad ($\theta_{\rm pert} = 0.1612\ldots$).

Another example of smooth process is the Gaussian walker satisfying $\frac{d^n X}{dt^n} = \eta(t)$, for $n \geq 2$ ($n = 1$ being the Markovian Brownian walker which is singular). The case $n = 2$ corresponds to a particle submitted to a random force, for which the persistence exponent is known to be $\theta = 1/4$ [29]. The two-time correlator when expressed in the fictitious time reads,

$$f(\tau) = \frac{3}{2}\exp(-|\tau|/2) - \frac{1}{2}\exp(-3|\tau|/2). \tag{69}$$

The IIA leads to $\theta_{\rm IIA} = 0.2647\ldots$, whereas $\theta_{\rm var} = 0.2857\ldots$, and $\theta_{\rm pert} = 0.3198\ldots$.

D. Perturbation around the action of a particle in a box

Let us briefly give a few applications of our expressions of Eq. (55,57,59).

As a simple test, they have been applied to the case of the Markovian walker with $\lambda = \theta = 1/2$, and $X_0 = 0$. The "variational" approach, which consists in taking the size of the box (or k_0) such that the first order perturbation vanishes leads to $2\theta_{\rm var} = 1.0074\ldots$. For the correlator given by Eq. (61), we found $\theta_{\rm var} = 1.4323\ldots$, in fair agreement with simulations and perturbative approaches around a Markovian process.

Finally, we have tested Eq. (59) in the case of the Brownian walker (for which $\lambda = 1/2$), for $X_0 \neq 0$. In this case, it is known that θ satisfies $\mathcal{D}_{2\theta}(X_0) = 0$ [27], where $\mathcal{D}_{2\theta}$ is a parabolic cylinder function. For $X_0 = 1/3$, (comparable to $\langle X^2 \rangle = f(0) = 1$, for $\lambda = 1/2$), we found $\theta_{\rm var} = 0.7032\ldots$, to be compared to the exact value $\theta = 0.6440\ldots$. Note that the perturbative expression of Eq. (45) leads to $\theta = 0.6330\ldots$.

IX. CONCLUSION

In this paper, we have stressed the importance of studying persistence for Gaussian stationary processes, as the calculation of θ for many physical systems can be often mapped on the persistence problem for this kind of process. We have then extended the perturbative

approach around a Markovian process, introduced in [10]. We have obtained resummed perturbative expressions (Eq. (34,37,51)) and a new self-consistent perturbative (or variational) expression for the persistence exponent (Eq. (40)). It seems that this variational result is more effective in reproducing numerical results, sometimes with impressive accuracy. We have also shown that all these expressions take a similar form as the alternative result of the IIA, which only applies to smooth processes. We have also given perturbative expressions for the X_0-level persistence exponent (Eq. (29,45)). Finally, we have shown that this type of perturbative approach is even more general, as the starting process around which we decide to perturb can be any Markovian process associated with a (preferably solvable) quantum problem. We have illustrated this point by explicitly deriving a variational expression for the X_0-level persistence exponent, when the starting quantum system is chosen to be a particle in a bounded box (Eq. (55,57,59)).

Finally, we conclude by pointing out that our perturbative and variational techniques have been useful in a wide variety of problems. This includes the calculation of the survival probability of a mobile particle in a fluctuating field [30], and the calculation of global persistence exponent in critical spin systems (to compute the order $\varepsilon^2 = (4-d)^2$ perturbative correction [11]), and for directed percolation [31].

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APPENDIX A

We want to compute $c_j = |\langle\hat{0}|x|\hat{j}\rangle|^2$, where $|\hat{j}\rangle$ is the jth eigenstate of the harmonic oscillator with an infinite barrier at the origin.

$$\langle\hat{j}|x\rangle = \sqrt{2}\langle 2j+1|x\rangle = \frac{\sqrt{2}}{\sqrt{2^{2j+1}(2j+1)!\sqrt{\pi}}} H_{2j+1}(x)\mathrm{e}^{-\frac{x^2}{2}}. \tag{70}$$

The extra factor $\sqrt{2}$ is due to the fact that $\langle\hat{j}|x\rangle$ is only defined on the interval $[0;+\infty]$, but should still be normalized. One then finds,

$$c_j = \frac{4}{\pi 2^{2j}(2j+1)!} I_j^2, \tag{71}$$

where $I_j = \int_0^{+\infty} x^2 \mathrm{e}^{-x^2} H_{2j+1}(x)\,dx$ can be readily calculated, using the properties $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, $H_n'(x) = 2nH_{n-1}(x)$ and $\int_0^{+\infty} \mathrm{e}^{-x^2} H_n(x)\,dx = H_{n-1}(0)$. This yields,

$$I_j = H_{2j}(0) + 4jH_{2j-2}(0) = (-1)^{j+1}\frac{(2j)!}{j!(2j-1)}, \tag{72}$$

which finally gives the result of the main text.

The c_j's satisfy the recursion relation,

$$\frac{c_{j+1}}{c_j} = \frac{(2j-1)^2}{(2j+3)(2j+2)}, \tag{73}$$

which shows that the generating function $f(x) = \sum_{j=0}^{+\infty} c_j x^j$ satisfies the hypergeometric differential equation $x(1-x)f'' + \frac{3}{2}f' - \frac{1}{4}f = 0$. The (unique) solution with $f(0) = c_0 = \frac{4}{\pi}$ is given by

$$f(x) = \frac{4}{\pi} F\left(-\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, x\right), \tag{74}$$

which yields the identities $\sum_{j=0}^{+\infty} c_j = f(1) = \frac{3}{2}$ and $\sum_{j=0}^{+\infty} jc_j = f'(1) = \frac{1}{4}$.

Now defining $U(\tau)$ as in Eq. (26) we get,

$$\frac{1}{2\lambda}(-\partial_\tau^2 + \lambda^2)^2 U(\tau) = \frac{\lambda^2}{2}\sum_{j=0}^{+\infty} C_j \exp(-2j\lambda|\tau|) = \frac{\lambda^2}{2} S[\exp(-2\lambda|\tau|)], \tag{75}$$

where $C_j = (4j^2-1)^2 c_j$ satisfies (using Eq. (73)),

$$\frac{C_{j+1}}{C_j} = \frac{j+\frac{3}{2}}{j+1}. \tag{76}$$

We recognize the recursion relation obeyed by the coefficients of the series expansion of the function $S(x) = \frac{4}{\pi}(1-x)^{-3/2}$, which leads to the final result of Eq. (28).

APPENDIX B

In this appendix, we write the general equation for E_0 and E_1, when the perturbation theory is applied to $\mathcal{S} = \mathcal{S}_{\mathrm{Q}}+\delta\mathcal{S}$. $\mathcal{S}_{\mathrm{Q}}$ is assumed to be the action associated with the quantum Hamiltonian H_0, with eigenenergies ε_l and eigenstates $|l\rangle$. The associated Hamiltonian with an infinite wall at the origin is H_1, with eigenenergies $\hat{\varepsilon}_l$ and eigenstates $|\hat{l}\rangle$. When the potential is symmetric with respect to $X = 0$, one simply has $\langle x|\hat{l}\rangle = \sqrt{2}\langle x|2l+1\rangle$, and $\hat{\varepsilon}_l = \varepsilon_{2l+1}$.

A. Equation for E_0

The lower "energy" E_0 is a functional of the inverse correlator $\hat{g}(\omega) = 1/\hat{f}(\omega)$:

$$E_0 = \varepsilon_0 + \lim_{\beta\to+\infty} \frac{1}{2\beta} \sum_{n=0}^{+\infty} (\hat{g}(\omega) - \hat{g}_0(\omega)) \cdot \langle 0|X(-\omega_n)X(\omega_n)|0\rangle, \tag{77}$$

where,

$$\hat{k}(\omega_n) = \hat{g}_0^{-1}(\omega_n) = \langle 0|X(-\omega_n)X(\omega_n)|0\rangle, \tag{78}$$

$$= \textstyle\int_0^{+\infty} 2\cos\omega_n\tau \sum_{l=0}^{+\infty} |\langle 0|x|l\rangle|^2 \mathrm{e}^{-(\varepsilon_l-\varepsilon_0)\tau} \mathrm{d}\tau = \displaystyle\sum_{l=1}^{+\infty} \frac{2d_l m_l^2}{d_l^2+\omega_n^2}, \tag{79}$$

with $d_l = \varepsilon_l - \varepsilon_0$, $m_l = |\langle 0|x|l\rangle|$ ($m_0 = 0$ due to the symmetry of the potential). Note that $\hat{k}$ is nothing more than the Fourier transform of the two-time correlation function of the position of the considered quantum particle.

Then, transforming the sum over Matsubara frequencies into an integral, one obtains:

$$E_0 = \varepsilon_0 + \frac{1}{2\pi} \int_0^{+\infty} \left(\frac{\hat{k}(x)}{\hat{f}(x)} - 1 \right) dx. \tag{80}$$

Due to the sum rule $\sum_{l=1}^{+\infty} 2d_l m_l^2 = 1$, valid for any Hamiltonian, the integrand tends to 0 as $x \to +\infty$.

In the text, we have considered two examples for the starting quantum correlator $\hat{k}$:

- Harmonic oscillator of frequency λ:
  $\varepsilon_l = \lambda(l+\frac{1}{2})$, and $d_l = \lambda l$; $m_l = (2\lambda)^{-1/2}\delta_{l,1}$; $\hat{k}(\omega) = (\omega^2+\lambda^2)^{-1}$, which directly leads to Eq. (17), obtained in section **IV** by expanding the exact result of Eq. (16).

- Particle in a box of width $2b = \pi/k_0$:
  $\varepsilon_l = \frac{1}{2}k_0^2(l+1)^2$, and $d_l = \frac{1}{2}k_0^2 l(l+2)$. After an elementary calculation involving the eigenstates of a particle in a box, we get,

$$m_l = (1-(-1)^l)\frac{1}{\pi k_0}\frac{4(l+1)}{l^2(l+2)^2}, \quad \hat{k}(\omega) = \frac{256}{\pi^2}\sum_{j=0}^{+\infty} \frac{a_j b_j}{k_0^4 b_j^2 + \omega^2}, \tag{81}$$

with,

$$a_j = \frac{2(j+1)^2}{(2j+1)^4(2j+3)^4}, \quad b_j = \frac{1}{2}(2j+1)(2j+3). \tag{82}$$

We then recover the expression of Eq. (55).

Note that the following sum rules have been used in the main text:

$$\frac{256}{\pi^2}\sum_{j=0}^{+\infty} a_j b_j = 1, \quad \frac{256}{\pi^2}\sum_{j=0}^{+\infty} a_j b_j^2 = \frac{1}{2}, \quad \frac{256}{\pi^2}\sum_{j=0}^{+\infty} \frac{a_j}{b_j} = \frac{5}{4} - \frac{\pi^2}{12}. \tag{83}$$

B. Equation for E_1

The "energy" E_1 is also a functional of the inverse correlator $\hat{g}(\omega) = 1/\hat{f}(\omega)$:

$$E_1 = \hat{\varepsilon}_0 + \lim_{\beta\to+\infty} \frac{1}{2\beta} \sum_{n=0}^{+\infty} (\hat{g}(\omega_n) - \hat{g}_0(\omega_n)) \cdot \langle \hat{0}|X(-\omega_n)X(\omega_n)|\hat{0}\rangle, \tag{84}$$

where, $\hat{k} = \hat{g}_0^{-1}$ has been defined in Eq. (78). Let us now introduce,

$$\begin{aligned} \hat{K}(\omega_n) &= \langle \hat{0}|X(-\omega_n)X(\omega_n)|\hat{0}\rangle, & (85)\\ &= \textstyle\int_0^{+\infty} 2\cos\omega_n\tau \sum_{l=0}^{+\infty} |\langle \hat{0}|x|\hat{l}\rangle|^2 \mathrm{e}^{-(\hat{\varepsilon}_l-\hat{\varepsilon}_0)\tau} \mathrm{d}\tau = \displaystyle\sum_{l=0}^{+\infty} \frac{2\hat{d}_l \hat{m}_l^2}{\hat{d}_l^2 + \omega_n^2}. & (86) \end{aligned}$$

$\hat{K}$ is the two-time correlator of the position of the quantum particle in the presence of the wall. As before, $\hat{d}_l = \hat{\varepsilon}_l - \hat{\varepsilon}_0$, $\hat{m}_l = |\langle \hat{0}|x|\hat{l}\rangle|$. Note that, contrary to the calculation for E_0, the $l = 0$ contribution in the sum above is non-zero so that, strictly speaking, this term should be written $2\pi\hat{m}_0^2\delta(\omega_n)$. This term has been written under this form in the main text.

Finally, the general expression of E_1 reads,

$$E_1 = \hat{\varepsilon}_0 + \frac{1}{2\pi}\int_0^{+\infty} \left(\frac{1}{\hat{f}(x)} - \frac{1}{\hat{k}(x)}\right) \hat{K}(x)\, dx. \tag{87}$$

We can again make this result more explicit for both considered quantum systems:

- Harmonic oscillator of frequency λ:
  $\hat{\varepsilon}_l = \lambda(2l + \frac{3}{2})$, $\hat{d}_l = 2\lambda l$, and $\hat{m}_l = \sqrt{c_l}$, which leads to the expressions of Eq. (22-25).

- Particle in a box of width $2b = \pi/k_0$:
  In this case, $\hat{K}$ is the Fourier transform of the two-time correlator of the position of a particle in a box of size $b = \pi/2k_0$. It is then clear using Eq. (81) that,

  $$\hat{K}(\omega) = \frac{1}{16}\hat{k}(\omega/4) + \frac{\pi^3}{8k_0^2}\delta(\omega), \tag{88}$$

  the extra Dirac peak coming from the fact that the operator X now has a finite average, as the particle belongs to the interval $[0, b]$. This immediately leads to the formula of Eq. (57), using $\hat{k}(0) = \frac{15-\pi^2}{12k_0^4}$.

When the constraint is $X \geq X_0$, with $X_0 = \eta b$ (as in section **VII**), the quantum particle now lives in a box of size $(1-\eta)b$, and the expression for $\hat{K}$ is changed accordingly, leading to,

$$\hat{K}(\omega) = \frac{(1-\eta)^4}{16}\hat{k}\left(\omega\frac{(1-\eta)^2}{4}\right) + \frac{\pi^3}{8k_0^2}(1+\eta)^2\delta\left(\omega\right). \tag{89}$$

This immediately leads to the result of Eq. (59).

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